Topology in Physics 2019 - lecture 15b

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In this course, we have seen many examples of how topology appears in physics. In recent years, however, the interaction between topology and physics has really become a two-way street. Physical ideas have often helped mathematicians in proving surprising theorems and in developing new structures and ideas. Today, we will briefly discuss a famous example of this: the relation between (topological) quantum field theories and *knot theory*.

15.1 Knot theory

Recall that in topology, mathematicians are interested in finding ***topological invariants***: maps $\phi : T \to X$ from a collection T of topological spaces to some other set X, such that the map is *invariant under homeomorphisms*: for $t_1, t_2 \in T$, ϕ must satisfy

$$t_1 \simeq t_2 \qquad \Longrightarrow \qquad \phi(t_1) = \phi(t_2). \tag{15.1}$$

where $\simeq$ denotes "homeomorphic to", that is: $t_1 \simeq t_2$ if there exists an invertible, continuous map between t_1 and t_2.

The target set X in this construction can take many different forms: a topological invariant can be a number, a group, a set or, as will be central in our discussion today, a polynomial. The set (or category) T that one is interested in can also vary: one can construct topological invariants for all topological spaces, for manifolds, for manifolds of a certain dimension, or as we will study here: for *knots*.

In mathematics, a knot does not have any end points: it can be viewed as an embedding

$$k : S^1 \to \mathbb{R}^3 \tag{15.2}$$

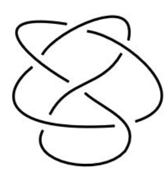
of the circle S^1 into three-dimensional space $\mathbb{R}^3$. Usually, these embeddings are taken to be smooth. Recall that the fact that k is an *embedding* means that $k(x) = k(y)$ implies $x = y$: the knot can not intersect itself.

Of course, defined in this way, a knot is not realy a topological space itself. One can cure this in different ways:

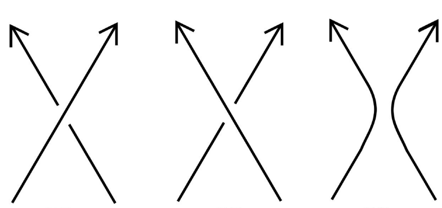
1. One can define the knot to be the topological space $K = \mathbb{R}^3 \setminus k(S^1)$, and study usch spaces up to homeomorphisms.
2. One can extend the concept of a topological invariant and homeomorphisms to embeddings. This requires some additional definitions but the construction is fully intuitive, so we will not discuss it in detail here.

The advantage of method (2) is that in this way, the knot keeps its *orientation*: traversing the image in the opposite direction leads to a formally different knot if we keep the data of the embedding, whereas simply removing the image of S^1 leads to exactly the same topological space. Whether we are interested in oriented or unoriented knots, the basic question remains: how can we figure out if two knots "are the same" in the sense that one can be continuously deformed into the other? Perhaps surprisingly, this is a very difficult mathematical problem!

Over the years, mathematicians have found many different ***knot invariants*** that help in addressing this problem. (Note that finding a knot invariant does not *solve* the problem in general: the arrow in 15.1 only points in one way, so if the invariant differs the knots differ, but not necessarily the other way around!) A large class of such invariants are constrcuted out of the ***projection*** of the knot onto the plane, that is: out of images[1] such as



or their oriented equivalents, with arrows drawn along the knot to picture the orientation. Here, we will discuss how one can use these projections to construct a *polynomial* that corresponds to the knot[2]. One way to do so, is to use a trick that was invented by John Conway in the 1960s: one uses a so-called ***skein relation***. A skein relation relates three knots that are almost the same, but differ at one single crossing. One lables the three types of this crossing, say for the oriented case for definiteness, by X_+, X_- and X_0 as follows:



[1]Of course, such images can be properly defined mathematically, but our aim here is to give a brief introduction, so we will not go into such details. There are many good books on knot theory where all the details can be found.

[2]In knot theory, a polynomial is often a Laurent polynomial: it can contain the variable also to negative powers, but always only contains a finite number of powers. Also, the variable itself is often written as a fractional power of another variable, e.g. the Jones polynomial is a Laurent polynomial in $t^{1/2}$.

and labels the corresponding knots by L_+, L_- and L_0. The skein relation then relates the polynomials for these three knots; for example, a well-known polynomial known as the *Alexander polynomial* $\nabla(L;z)$ satisfies

$$\nabla(L_+;z) - \nabla(L_-;z) = z\nabla(L_0;z). \tag{15.3}$$

Conway proved that if one now also assigns a value to the trivial projection U of the *unknot*, that is: a projection which is an ordinary circle without any crossings, for example

$$\nabla(U;z) = 1 \tag{15.4}$$

then

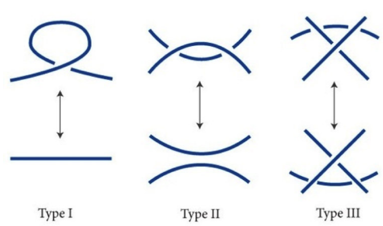
1. By repeatedly using skein relations, one can express the polynomial for *any* knot projection in terms of the value for the unknot projection,
2. There may be several ways to reduce a known projection to the unknot projection, but the resulting polynomial does not depend on the construction one chooses,
3. There may be several projections for a single knot (for example, one can also project the unknot in such a way that it has several crossings in its image), but the resulting polynomial also does not depend on the projection.

That is, the Alexander polynomial is truly a knot invariant!

One technical detail should be mentioned here: by using the skein relation, one may "disconnect" a knot into an object that has several S^1-components. These objects are known as ***links***, and one can easily enlarge the concept of a knot invariant to a link invariant.

To prove that a polynomial such as the Alexander polynomial is indeed a knot invariant, there is a very useful

Theorem: *Any two projections of homeomorphic knots can be related by repeatedly applying the following three moves:*



The above moves are known as the ***Reidemeister moves***; the theorem was proved in the 1920s by Kurt Reidemeister and independently by James Alexander and Garland Briggs.

As a result of the theorem, if one can prove that a certain knot polynomial is invariant under the three Reidemeister moves, one has immediately proven that it is a knot invariant. This leads to rather easy proofs: for example, using this method it is a fun exercise to show that the Jones polynomial that we will encounter in the next section is also a knot invariant.

15.2 Chern-Simons theory and the Jones polynomial

How is all of the above related to physics? This was discovered by physicist Edward Witten in 1989. Witten studied Chern-Simons theory, and discovered that it can be used to (re-)construct a famous knot invariant.

Recall that Chern-Simons theory, discussed in lecture 8, is a quantum field theory for a Lie-algebra valued gauge field (that is, a connection) A on a three-manifold M. The action of the theory – at least in local form – can be written as

$$S_{CS} = \frac{k}{4\pi} \int_M \mathrm{Tr}\ (A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \tag{15.5}$$

Recall that for the theory to be well-defined globally, k must be an integer. Since the action above is fully written in terms of differential forms and connections, it does not depend on either a choice of coordinates or a metric on M. That is: it is a topological invariant of M. Moreover, we saw in lecture 8 that Chern-Simons theory has observables that are also topologically invariant: given a closed loop $\gamma : [0,1] \to M$, one can construct the ***Wilson loop***

$$W_\gamma(k) = \left\langle \mathrm{Tr}\ \mathcal{P} \exp \left(- \int_\gamma (A_\mu)^i{}_j dx^\mu \right) \right\rangle, \tag{15.6}$$

which only depends on three things:

1. The topology of M, which we can take to be e.g. $\mathbb{R}^3$ or, as is also often done in knot theory, S^3,
2. The topology of the image of γ,
3. The value of the coupling constant k.

That is: once we fix the topology of M, the resulting function of k is a knot invarant! Witten managed to prove that for gauge group $SU(2)$, this invariant was actually not a new one: if one uses the variable

$$t = \exp\left(\frac{2\pi i}{k+2}\right) \tag{15.7}$$

then $W_\gamma(k) \equiv J(t)$ turned out to be a *polynomial* in t, and Witten was able to show that this polynomial moreover satisfied the skein relation

$$t^{-1}\ J(L_+;t) - t\ J(L_-;t) = (t^{1/2} - t^{-1/2})\ J(L_0;t) \tag{15.8}$$

This skein relation was well-known: it was known to be the skein relation of the ***Jones polynomial***, a knot invariant that had been constructed by Vaughan Jones five years earlier, in 1984, using completely different methods.

The details of the proof of the skein relation are unfortunately too complicated to discuss here, but the idea is very "physical" in nature: Witten uses a cutting-and-pasting argument, where M is constructed out of two halves, M_1 and M_2. The crossing to which one wants to apply the skein relation is near the cutting surface and the knots L_+, L_- and L_0 can be obtained by glueing the halves of M in three different ways. Now M_1 and M_2 have a boundary, and therefore the expectation value of the Wilson loop in each of them is a function of the field configuration at this boundary. That is: it is a quantum state (wave function) on this boundary! After gluing, the Wilson loops for the three different knots can therefore be computed as three inner products in the Hilbert space of quantum states on this boundary. Witten proves that this Hilbert space is in fact two-dimensional, so that there must be a linear relation between the three inner products. He then shows that this relation is precisely the skein relation above.

Witten and others extended the above ideas to other gauge groups and to other representation, which led to a whole host of constructions of knot invariants – some known before, and some new. In this way, quantum field theory had an important impact on knot theory (and more generally, topology, as similar ideas kan of course also be used for other topological spaces), opening a line of research that is still very actively pursued today.